

zeta-functions and determinants

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We consider in detail the quantum-mechanical problem associated with the motion of a one-dimensional particle under the action of the double-well potential. Our main tool will be the euclidean (imaginary time) version of the path-integral method. Once we perform the Wick rotation, the euclidean equation of motion is the same as the usual one for the point particle in real time, except that the potential at issue is turned upside down. In doing so, our double-well potential becomes a two-humped potential. As required by the semiclassical approximation we may study the quadratic fluctuations over the instanton which represents in this context the localised finite-action solutions of the euclidean equation of motion. The determinants of the quadratic differential operators are evaluated by means of the zeta-function method. We write in closed form the eigenfunctions as well as the energy eigenvalues corresponding to such operators by using the shape-invariance symmetry. The effect of the multi-instantons configurations is also included in this approach.

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I. INTRODUCTION.

The semiclassical approximation to the euclidean path-integral represents one of the most valuable tools in quantum theory. By expanding around topological configurations one can obtain information about the tunneling phenomenon in physical systems ranging from non-relativistic one-particle quantum mechanics to relativistic gauge field theories. The basis of this approach relies on the so-called instanton which represents a localised finite-action solution of the euclidean equation of motion where the time variable is essentially imaginary. To sum up, we first find the appropriate classical configuration and subsequently evaluate the quadratic fluctuations. The functional integration is solved in terms of the gaussian scheme except for the zero-modes which appear by virtue of the translational invariance of the system. It is customarily assumed the existence of collective coordinates so that as a matter of fact one performs the gaussian integration in the directions orthogonal to the zero-modes.

To illustrate the usual instanton method in a simple model we resort to the double-well potential as first considered by Polyakov [1] in his pioneering work on the subject. The famous problem of the level splittings is analyzed in terms of the instanton calculus which represents by itself an alternative to the WKB technique. From a physical point of view the result can be summarized as follows: the quantum mechanical amplitude consists on an exponential of the classical action of the topological configuration multiplied by a fluctuation factor which takes into account the inverse square root of the determinant of harmonic eigenmodes. In this paper we will not try to expose new fundamental developments on the matter but rather focus the attention to the explicit evaluation of the determinants at issue.

By analogy with the finite-dimensional case one can suspect that a functional determinant includes an infinite product of eigenvalues. As expected this expression will be highly divergent but in any case we may regularize it by means of the ratio of two determinants. When the eigenfunctions as well as the energy eigenvalues of the quadratic operators are known in closed form the zeta-function method serves to evaluate this quantity in a sys-

tematic way [2]. For the double-well potential it proves convenient to take advantage of the shape-invariance symmetry so that the only tools we need are supersymmetric quantum mechanics and the physics of the free-particle.

In order to obtain an accurate description of the level splitting for a double-well potential (or double-humped once the potential is turned upside down) we introduce a chain of instantons and anti-instantons. If the distance between the centers of such configurations is large the interaction can be neglected. This framework represents the dilute gas approximation in quantum theory. The article is organised as follows. First of all we expose in detail the instantonic approach in non-relativistic one-dimensional particle mechanics. To illustrate the procedure in a simple but relevant context we consider the aforementioned double-well potential. We keep for the last section the analysis of the effect of the multi-instantons configurations.

II. INSTANTONS IN PARTICLE MECHANICS.

In this section we describe the instanton calculus for the one-dimensional spinless particle as can be found for instance in [3]. For reference we assume that our particle moves under the effect of a confining potential $V(x)$ so that the quantum model exhibits a pure discrete spectrum of energy eigenvalues. Without loss of generality we choose the origin of the energy such that the minimum (minima) of the potential satisfy $V(x) = 0$. Now taking the mass of the particle equal to unity, i.e. $m = 1$, the lagrangian which governs the behaviour of the model corresponds to

$$L = \frac{1}{2} \left(\frac{dx}{dt} \right)^2 - V(x) \quad (1)$$

If the particle is located at the initial time $t_i = -T/2$ at the point x_i while one finds it when $t_f = T/2$ at the point x_f , the functional version of the non-relativistic quantum mechanics allows us to express the transition amplitude in terms of a sum over all paths joining the world points with coordinates $(-T/2, x_i)$ and $(T/2, x_f)$. The main physical

ingredient consists in the way in which we weight the trajectories according to the imaginary exponential of the action. At this point it proves convenient to write the action S starting from the lagrangian L , i.e.

$$S = \int_{-T/2}^{T/2} L(x, \dot{x}) dt \quad (2)$$

so that the transition amplitude is

$$\langle x_f | \exp(-iHT) | x_i \rangle = N(T) \int [dx] \exp iS[x(t)] \quad (3)$$

It may be worth spelling out that H represents the standard hamiltonian of the model while the symbol $[dx]$ serves to indicate the integration over all functions which fulfill the aforementioned boundary conditions. The factor $N(T)$ will be adjusted later on to make the whole integral finite and suitable normalised. As the hamiltonian H yields a pure discrete spectrum of energy eigenvalues, namely

$$H|n\rangle = E_n|n\rangle \quad (4)$$

we have that

$$\langle x_f | \exp(-iHT) | x_i \rangle = \sum_n \exp(-iE_n T) \langle x_f | n \rangle \langle n | x_i \rangle \quad (5)$$

As this last expression is difficult to handle due to the oscillating character of the exponential, we perform the transition to the imaginary time. Making the substitution $t \rightarrow -i\tau$, known in the literature as the Wick rotation, one can take advantage of the euclidean formalism where

$$iS[x(t)] \rightarrow \int_{-T/2}^{T/2} \left[-\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 - V(x) \right] d\tau \quad (6)$$

Now it remains to identify the so-called euclidean action S_e , i.e.

$$S_e = \int_{-T/2}^{T/2} \left[\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right] d\tau \quad (7)$$

The euclidean version of (3) corresponds to

$$\langle x_f | \exp(-HT) | x_i \rangle = N(T) \int [dx] \exp -S_e[x(\tau)] \quad (8)$$

In order to explain the meaning of the semiclassical approximation we consider a function $x_c(\tau)$ satisfying the boundary conditions at issue. Now we expand a general $x(\tau)$ with the same boundary conditions according to

$$x(\tau) = x_c(\tau) + \sum_j c_j x_j(\tau) \quad (9)$$

where $x_j(\tau)$ constitute a complete set of orthonormal functions

$$\int_{-T/2}^{T/2} x_j(\tau) x_k(\tau) d\tau = \delta_{jk} \quad (10)$$

vanishing at the boundary

$$x_j(\pm T/2) = 0 \quad (11)$$

In addition the $[dx]$ can be chosen as

$$[dx] = \prod_j \frac{dc_j}{\sqrt{2\pi}} \quad (12)$$

The semiclassical approximation takes for granted that $x_c(\tau)$ represents a stationary point of the euclidean action, thus satisfying the equation

$$\frac{d^2x}{d\tau^2} = V'(x) \quad (13)$$

where the prime denotes as usual the derivative with respect to the spatial coordinate. We recognize in (13) the euclidean equation of motion for our particle once the potential has been turned upside down. Now the crucial feature is the analysis of the second variation of the action with respect to $x_c(\tau)$. (There is no linear variation due to the stationary point character of $x_c(\tau)$). In doing so we find that

$$S_e[x_c(\tau) + \delta x(\tau)] = S_{eo} + \int_{-T/2}^{T/2} \delta x \left[-\frac{1}{2} \frac{d^2}{d\tau^2} \delta x + \frac{1}{2} V''[x_c(\tau)] \delta x \right] d\tau \quad (14)$$

being S_{eo} the classical action associated with the configuration $x_c(\tau)$. The procedure becomes much more accesible in terms of a complete set of eigenfunctions (eigenvalues) of the so-called stability equation

$$-\frac{d^2}{d\tau^2}v_j(\tau) + V''[x_c(\tau)]v_j(\tau) = \epsilon_j v_j(\tau) \quad (15)$$

so that the family just anticipated in (10) is obtained by means of (15). The problem becomes diagonal in this new scheme so that

$$S_e = S_{eo} + \frac{1}{2} \sum_j c_j^2 \epsilon_j \quad (16)$$

Taking into account the standard gaussian integral, i.e.

$$\int_{-\infty}^{\infty} \exp(-\lambda z^2/2) dz = \sqrt{\frac{2\pi}{\lambda}} \quad (\lambda > 0) \quad (17)$$

the euclidean transition amplitude reduces to

$$\langle x_f | \exp(-HT) | x_i \rangle = N(T) \exp(-S_{eo}) \prod_j \epsilon_j^{-1/2} \quad (18)$$

Almost from the very beginning of the subject the product of eigenvalues has been written as

$$\prod_j \epsilon_j^{-1/2} = \left\{ Det \left[-\frac{d^2}{d\tau^2} + V''[x_c(\tau)] \right] \right\}^{-1/2} \quad (19)$$

a notation which obviously comes from the finite-dimensional case. It should be emphasized that by now we assume that all the eigenvalues ϵ_j are positive to avoid problems of definition for (19) due to the existence of zero-modes (more on this later). The physical meaning of (18) can be summarized as follows: the transition amplitude consists on an exponential of the classical action of the configuration $x_c(\tau)$ at issue multiplied by a factor which takes into account the inverse square root of the determinant of a second order differential operator.

To close this section we focus the interest on the explicit evaluation of the euclidean transition amplitude for the well-grounded problem of the harmonic oscillator. To be precise we consider boundary conditions such that $x_i = x_f = 0$. The purpose of the calculation is twofold. First of all we find physical arguments to fix the factor $N(T)$. On the other hand the harmonic oscillator acts as reference to deal with the ratio of determinants when we go to more complicated systems. Let us explore therefore the case where

$$V(x) = \frac{\nu^2}{2} x^2 \quad (20)$$

so that $V''(x=0) = \nu^2$. If the potential is turned upside down the euclidean equation of motion dictates that the only dynamical possibility starting from $x_i = 0$ at $t_i = -T/2$ is $x_c(\tau) = 0$ whenever we require also that $x_f = 0$ when $t_f = T/2$. Otherwise the particle scapes towards plus or minus infinity. As the classical action associated with this trivial path vanishes the semiclassical approximation reduces itself to

$$\langle x_f = 0 | \exp(-H_{ho}T) | x_i = 0 \rangle = N(T) \left\{ \text{Det} \left[-\frac{d^2}{d\tau^2} + \nu^2 \right] \right\}^{-1/2} \quad (21)$$

For the explicit evaluation of (21) we follow the pattern exposed in [4]. To start from scratch we write a complete set of eigenfunctions of the quadratic operator whose determinant we desire to compute. To be precise

$$x_j = \cos \lambda \tau \quad (22)$$

$$\tilde{x}_j = \sin \lambda \tau \quad (23)$$

Once we add the boundary conditions (see (11)) the eigenvalues of energy ϵ_j which correspond to (22) and (23) are of the form

$$\epsilon_j = \frac{j^2 \pi^2}{T^2} + \nu^2 \quad (j = 1, 2, \dots) \quad (24)$$

Now the principal tool for extracting the physical information hidden in the product of eigenvalues is the factorization of the determinant, namely

$$N(T) \left\{ \text{Det} \left[-\frac{d^2}{d\tau^2} + \nu^2 \right] \right\}^{-1/2} = N(T) \left[\prod_{j=1}^{\infty} \frac{j^2 \pi^2}{T^2} \right]^{-1/2} \left[\prod_{j=1}^{\infty} \left(1 + \frac{\nu^2 T^2}{j^2 \pi^2} \right) \right]^{-1/2} \quad (25)$$

As can be seen easily the two first factors represent the euclidean transition amplitude for the free-particle so that

$$N(T) \left[\prod_{j=1}^{\infty} \frac{j^2 \pi^2}{T^2} \right]^{-1/2} = \langle x_f = 0 | \exp(-H_o T) | x_i = 0 \rangle \quad (26)$$

where H_o corresponds to

$$H_o = -\frac{d^2}{d\tau^2} \quad (27)$$

Now it suffices to resort to a set of plane waves labeled by $|k\rangle$ to obtain that

$$N(T) \left[\prod_{j=1}^{\infty} \frac{j^2 \pi^2}{T^2} \right]^{-1/2} = \int_{-\infty}^{\infty} \exp(-k^2 T/2) \frac{dk}{2\pi} \quad (28)$$

Using again the formula (17) we find

$$N(T) \left[\prod_{j=1}^{\infty} \frac{j^2 \pi^2}{T^2} \right]^{-1/2} = \frac{1}{\sqrt{2\pi T}} \quad (29)$$

The last factor of (25) can be transformed into a more suitable expression according to [5]

$$\pi z \prod_{j=1}^{\infty} \left(1 + \frac{z^2}{j^2} \right) = \sinh \pi z \quad (30)$$

where in our case it happens that

$$z = \frac{\nu T}{\pi} \quad (31)$$

With all these partial results at hand the final form for the amplitude written in (21) should be

$$\langle x_f = 0 | \exp(-H_{ho}T) | x_i = 0 \rangle = \left(\frac{\nu}{\pi} \right)^{1/2} (2 \sinh \nu T)^{-1/2} \quad (32)$$

For reference we point out the behaviour of (32) when considering the limit $T \rightarrow \infty$ since this final step is crucial for instance in the analysis of the double-well potential to get valuable information about the splitting. To sum up

$$\langle x_f = 0 | \exp(-H_{ho}T) | x_i = 0 \rangle \longrightarrow \left(\frac{\nu}{\pi} \right)^{1/2} \exp(-\nu T/2) \left[1 + \frac{1}{2} \exp(-2\nu T) + \dots \right] \quad (33)$$

Although it is a simple matter to check in (32) how this semiclassical approximation is exact for the harmonic oscillator we prefer to go to more relevant physical systems where the euclidean version of the path-integral represents probably the most elegant tool to study the tunneling phenomenon.

III. THE DOUBLE-WELL POTENTIAL.

Once we have understood the main properties of the instanton calculus for the one-dimensional particle we proceed to examine the method in a concrete example. For such a purpose one needs a particle moving in a potential well which has a set of degenerate minima. No matter what the specific details of the potential, it is the case that the euclidean equation of motion exhibits topologically nontrivial solutions with finite-action. As anticipated in the previous section these solutions connect adjoining minima of the potential at issue and make the transition through a region classically forbidden. Although a periodic potential should be an excellent benchmark in this context we consider now the double-well potential $V(x)$ given by

$$V(x) = \frac{\omega^2}{8}(x^2 - 1)^2 \tag{34}$$

From a classical point of view there are two minima located at $x_- = -1$ and $x_+ = 1$ as expected when dealing with a potential which enjoys the discrete symmetry $x \rightarrow -x$. One could just as well assume that $\omega^2 \gg 1$ so that the barrier is high enough to decompose the system into a sum of two independent harmonic oscillators widely separated from each other. Then the lowest state of the model as a whole has a twofold degeneracy. Qualitatively speaking the particle lives in the right well or in the left one while executes small oscillations around the points $x_- = -1$ or $x_+ = 1$. For a finite barrier however, the particle in either of the two wells has a non-vanishing transition amplitude to tunnel into the other well so that the individual wave functions for left-hand and right-hand harmonic oscillators will mix. To sum up, the discrete symmetry $x \rightarrow -x$ is not spontaneously broken since the expectation value of the coordinate x evaluated for the ground state is zero as expected for a wave function which makes an even superposition of wave functions centered at either of the two left and right wells.

Now we can carry things further and discuss how the tunneling phenomenon emerges in the euclidean framework. The analysis is based on the existence of a topological configuration

which connects the two minima located at $x = \pm 1$. The solution of the euclidean equation of motion which makes the connection is precisely the instanton according to the name first coined in the mid-seventies. Following the conventional wisdom the instanton interpolates between adjoining minima of the potential and lifts the degeneracy of the classical vacua. The instanton only appears once the potential is turned upside down. In doing so the two-well model transforms into a double-humped potential. At first glance it seems physically relevant to take into account the transition between the points $x_- = -1$ and $x_+ = 1$ whenever we have previously managed to obtain the topological solution in closed form.

Therefore we are interested in a trajectory which starts at $t_i = -T/2$ from the point $x_i = -1$ and reaches $x_f = 1$ when $t_f = T/2$. It is customarily assumed that $T \rightarrow \infty$ since the solution of the problem for finite T is much more complicated (more on this later). This movement, where the particle itself takes an infinite time to climb up (or drop from) the small piece next to the top of the two potential mountains, can be understood in terms of the conservation law of the euclidean energy once the double-well is just reversed. Simply put, we solve the problem by integration of a first-order differential equation instead of resorting to the conventional euclidean equation of motion written in (13). As the euclidean energy is equal to zero for the motion of our particle, the form of the topological solution is quite simple, namely

$$x_c(\tau) = \tanh \frac{\omega(\tau - \tau_c)}{2} \quad (35)$$

where the parameter τ_c indicates the point at which the instanton makes the jump. Of course the so-called antiinstanton is obtained by means of the transformation $\tau \rightarrow -\tau$ and allows the connection between $x_i = 1$ and $x_f = -1$. Now we can compute the action associated with $x_c(\tau)$ according to (7) so that

$$S_{eo} = \frac{2\omega}{3} \quad (36)$$

It may be interesting at this point to remind that actually we need configurations for which x is equal to ± 1 at large but finite values $\tau = \pm T/2$. However the topological solution

written in (35) reaches the points ± 1 at infinite times. Fortunately the difference is so small that it can be ignored since at the end of the procedure we are interested precisely in the large T limit. As required by the semiclassical approximation we must calculate the quadratic fluctuations correction once the classical contribution is known. The question we wish to address now is the analysis of the transition amplitude between $x_i = -1$ and $x_f = 1$ taking as reference the harmonic oscillator derived from

$$V''(x = \pm 1) = \omega^2 \quad (37)$$

Our description takes over

$$\begin{aligned} \langle x_f = 1 | \exp(-HT) | x_i = -1 \rangle = N(T) \left\{ \text{Det} \left[-\frac{d^2}{d\tau^2} + \omega^2 \right] \right\}^{-1/2} \\ \left\{ \frac{\text{Det} [-(d^2/d\tau^2) + V''[x_c(\tau)]]}{\text{Det} [-(d^2/d\tau^2) + \omega^2]} \right\}^{-1/2} \exp(-S_{eo}) \end{aligned} \quad (38)$$

where we have multiplied and divided by the determinant for the harmonic oscillator precisely. Incorporating the explicit form of $x_c(\tau)$, where we fix by now $\tau_c = 0$, the stability equation reads

$$-\frac{d^2}{d\tau^2} x_j(\tau) + \left[\omega^2 - \frac{3\omega^2}{2 \cosh^2(\omega\tau/2)} \right] x_j(\tau) = \epsilon_j x_j(\tau) \quad (39)$$

and corresponds to a Schrodinger's equation with Posch-Teller potential. Although the problem can be studied in more conventional forms we prefer to do that by means of the shape-invariance symmetry [6]. Ultimately the only tools we need are supersymmetric quantum mechanics and the free particle. (The details can be consulted in Appendix A). Among other things one finds a zero-mode $x_o(\tau)$ which in principle could jeopardize the evaluation of the determinant. However this eigenvalue $\epsilon_o = 0$ comes by no surprise since it reflects the translational invariance of the system. In other words, there is one direction in the functional space of the second variations which is incapable of changing the action. As a matter of fact one can discover the existence of a zero-mode starting from (13). It suffices an additional derivative with respect to τ to find that

$$x_o(\tau) = \frac{1}{\sqrt{S_{eo}}} \frac{dx_c}{d\tau} \quad (40)$$

is just the solution of (15) with $\epsilon_o = 0$. Notice the normalization of $x_o(\tau)$ which is due precisely to the zero euclidean energy condition for $x_c(\tau)$. The way out of this apparent cul-de-sac is simple. The integration over c_o (see (9)) becomes equivalent to the integration over the center of the instanton τ_c . To fix the jacobian of the transformation involved we take a first change such that

$$\Delta x(\tau) = x_o(\tau) \Delta c_o \quad (41)$$

According to the general expression written in (9) we find that under a shift $\Delta\tau_c$ the effect should be

$$\Delta x(\tau) = -\sqrt{S_{eo}} x_o(\tau) \Delta\tau_c \quad (42)$$

Now the identification between (41) and (42) yields

$$dc_o = \sqrt{S_{eo}} d\tau_c \quad (43)$$

where the minus sign disappears since what matters is the modulus of the jacobian at issue. In doing so we have that

$$\left\{ \frac{Det[-(d^2/d\tau^2) + V''[x_c(\tau)]]}{Det[-(d^2/d\tau^2) + \omega^2]} \right\}^{-1/2} = \left\{ \frac{Det'[-(d^2/d\tau^2) + V''[x_c(\tau)]]}{Det[-(d^2/d\tau^2) + \omega^2]} \right\}^{-1/2} \sqrt{\frac{S_{eo}}{2\pi}} d\tau_c \quad (44)$$

where Det' stands for the so-called reduced determinant once the zero-mode has been removed.

Different methods can be found in the literature to find the ratio of determinants written in (44). One takes firstly the system enclosed in a box of length T , thus avoiding the subtleties associated with the continuous spectrum [3]. At the end of the whole procedure it suffices the limit $T \rightarrow \infty$ to achieve a meaningful result. However we prefer from the very beginning

to work in open space and evaluate the ratio of determinants by using the zeta-function method. (The interested reader can find in the Appendix B a sketch of this technique as tailored to our needs). Going back to (39) and (44), it suffices to make the change of variable

$$z = \frac{\omega\tau}{2} \quad (45)$$

to recognize the presence of the ratio of determinants (see Appendix B)

$$Q_2 = \frac{Det' O_2}{Det P_2} \quad (46)$$

togheter with a global factor β given by

$$\beta = \frac{\omega^2}{4} \quad (47)$$

As regards the non-zero spectrum of the operator O_2 we have a discrete level with $E_1 = 3$ while the energy of the scattering states corresponds to

$$E_k = k^2 + 4 \quad (48)$$

for eigenfunctions derived from the plane waves according to

$$\phi_{2,k}(z) = \frac{A_2^\dagger(z)}{\sqrt{k^2 + 4}} \frac{A_1^\dagger(z)}{\sqrt{k^2 + 1}} \left[\frac{\exp(ikz)}{\sqrt{2\pi}} \right] \quad (49)$$

The explicit form of the operators $A_2^\dagger(z)$ and $A_1^\dagger(z)$, namely

$$A_1^\dagger = -\frac{d}{dz} + \tanh z \quad (50)$$

$$A_2^\dagger = -\frac{d}{dz} + 2 \tanh z \quad (51)$$

allows us to find that

$$\phi_{2,k}(z) = \left(-k^2 + 2 - \frac{3}{\cosh^2 z} - 3ik \tanh z \right) \frac{\exp(ikz)}{\sqrt{2\pi} \sqrt{k^2 + 4} \sqrt{k^2 + 1}} \quad (52)$$

As a natural consequence of (52) the regularized spectral density $\rho_r(k)$ of the problem reads

$$\rho_r(k) = -\frac{3(k^2 + 2)}{\pi(k^2 + 4)(k^2 + 1)} \quad (53)$$

In this scheme the suitable zeta-function $\zeta_r(s)$ to evaluate the ratio of determinants of (44) can be understood as

$$\zeta_r(s) = \zeta_{O_2}(s) - \zeta_{P_2}(s) \quad (54)$$

This information is sufficient to write

$$\zeta_r(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mu^{s-1} d\mu \left[\exp(-3\mu) - \frac{3}{\pi} \int_{-\infty}^\infty \frac{(k^2 + 2) \exp[-(k^2 + 4)\mu]}{(k^2 + 4)(k^2 + 1)} dk \right] \quad (55)$$

which, in turn, gives rise to [5]

$$\zeta_r(s) = \frac{1}{3^s} - \frac{3}{\pi} \int_{-\infty}^\infty \frac{(k^2 + 2)}{(k^2 + 1)(k^2 + 4)^{s+1}} dk \quad (56)$$

Breaking the integral of (56) into more simple components it is not difficult to obtain $\zeta_r(s)$ in terms of Gamma and Hypergeometric Functions, namely [5]

$$\zeta_r(s) = \frac{1}{3^s} - \frac{3}{\sqrt{\pi}} \frac{1}{2^{2s+1}} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s + 1)} - \frac{3}{\sqrt{\pi}} \frac{1}{2^{2s+3}} \frac{\Gamma(s + \frac{3}{2})}{\Gamma(s + 2)} F\left(1, s + \frac{3}{2}, s + 2, \frac{3}{4}\right) \quad (57)$$

Taking into account that [7]

$$F\left(1, \frac{3}{2}, 2, \frac{3}{4}\right) = \frac{8}{3} \quad (58)$$

the evaluation of $\zeta_r(s)$ at $s = 0$ yields

$$\zeta_r(0) = -1 \quad (59)$$

Let us come down to the concrete details which allow the evaluation of $\zeta'_r(0)$. Our approach provides a result of the form

$$\zeta'_r(0) = -\ln 3 + 8 \ln 2 - \frac{1}{2} - \frac{3}{16} F'(s + \frac{3}{2}, 1, s + 2, \frac{3}{4})|_{s=0} \quad (60)$$

once one incorporates that [7]

$$\Gamma'(\frac{1}{2}) = -\sqrt{\pi} (\gamma + 2 \ln 2) \quad (61)$$

$$\Gamma'(1) = -\gamma \quad (62)$$

$$\Gamma'(\frac{3}{2}) = -\sqrt{\pi} (\frac{\gamma}{2} + \ln 2 - 1) \quad (63)$$

$$\Gamma'(2) = -\gamma + 1 \quad (64)$$

where γ is the Euler's constant. In order to obtain the derivative of the Hypergeometric Function at issue we resort to the so-called integral representation, namely [7]

$$F(s + \frac{3}{2}, 1, s + 2, \frac{3}{4}) = \frac{\Gamma(s + 2)}{\Gamma(1) \Gamma(s + 1)} \int_0^1 (1 - t)^s \left(1 - \frac{3t}{4}\right)^{-s - \frac{3}{2}} dt \quad (65)$$

so that

$$F'(s + \frac{3}{2}, 1, s + 2, \frac{3}{4})|_{s=0} = I_1 + I_2 + I_3 \quad (66)$$

for integrals of the form

$$I_1 = \int_0^1 \left(1 - \frac{3t}{4}\right)^{-\frac{3}{2}} dt \quad (67)$$

$$I_2 = \int_0^1 \left(1 - \frac{3t}{4}\right)^{-\frac{3}{2}} \ln(1 - t) dt \quad (68)$$

$$I_3 = - \int_0^1 \left(1 - \frac{3t}{4}\right)^{-\frac{3}{2}} \ln\left(1 - \frac{3t}{4}\right) dt \quad (69)$$

If we have that

$$I_1 = \frac{8}{3} \quad (70)$$

$$I_2 = \frac{32}{3} \ln \frac{2}{3} \quad (71)$$

$$I_3 = -\frac{16}{3} + \frac{32}{3} \ln 2 \quad (72)$$

then $\zeta'_r(0)$ is obviously

$$\zeta_r'(0) = 4 \ln 2 + \ln 3 \quad (73)$$

Under these conditions we can write finally the ratio of determinants R which appear in (44), namely

$$R = \frac{Det' [-(d^2/d\tau^2) + V''[x_c(\tau)]]}{Det [-(d^2/d\tau^2) + \omega^2]} \quad (74)$$

In a nutshell

$$R = \frac{1}{12\omega^2} \quad (75)$$

Combining now the results obtained along this section we write the transition amplitude anticipated in (38), i.e.

$$\begin{aligned} &< x_f = 1 | \exp(-HT) | x_i = -1 > = \\ &\left(\frac{\omega}{\pi}\right)^{1/2} (2 \sinh \omega T)^{-1/2} \sqrt{S_{eo}} \sqrt{\frac{6}{\pi}} \exp(-S_{eo}) \omega d\tau_c \end{aligned} \quad (76)$$

Apart from the first factor, which represents the contribution of the harmonic oscillator of reference, we have a transition amplitude just depending on the point τ_c at which the instanton makes the jump. In accordance with the time interval T the result seems plausible whenever

$$\sqrt{S_{eo}} \sqrt{\frac{6}{\pi}} \exp(-S_{eo}) \omega T \ll 1 \quad (77)$$

a nonsense condition if T is large enough. But precisely in this regime we can accommodate configurations constructed of instantons and antiinstantons which mimic the behaviour of a trajectory just derived from the euclidean equation of motion. In doing so we get an additional bonus since the integration over the parameter τ_c is feasible. As a matter of fact the level splitting formula is deduced from the complete transition amplitude once we take into account the multi-instanton configurations. This serves to exhibit the instanton calculus as an alternative to the WKB approximation in this context.

IV. THE DILUTE-GAS APPROXIMATION.

Although all the above calculations were carried out over a single instanton, it remains to identify the hypothetical contributions which incorporate the effect of a string of instantons and antiinstantons along the τ axis. It is customarily assumed that these combinations of topological solutions represent no strong deviations of the trajectories just derived from the euclidean equation of motion without any kind of approximation. For background suppose we have j instantons and antiinstantons centered at points τ_1, \dots, τ_j whenever

$$-\frac{T}{2} < \tau_1 < \dots < \tau_j < \frac{T}{2} \quad (78)$$

Being narrow enough the regions where the instantons (antiinstantons) make the jump, the action of the proposed configuration is almost extremal. We can carry things further and assume that the action of the string of instantons and antiinstantons is given by the sum of the j individual actions. This scheme is well-known in the literature where appears with the name of dilute gas approximation [8].

It should be emphasized at this point that now we can compute amplitudes with closed paths with $x_i = -1 = x_f$ for instance, so that the action at issue S_t will be an even multiple of the single instanton action, i.e.

$$S_t \approx 2j S_{eo} \quad (79)$$

As expected the complete transition amplitude between $x_i = -1$ and $x_i = 1$ incorporates the contribution

$$S_t \approx (2j + 1) S_{eo} \quad (80)$$

In addition the translational degrees of freedom of the separated j instantons and antiinstantons lead to an integral of the form

$$\int_{-T/2}^{T/2} \omega d\tau_j \int_{-T/2}^{\tau_j} \omega d\tau_{j-1} \dots \int_{-T/2}^{\tau_2} \omega d\tau_1 = \frac{(\omega T)^j}{j!} \quad (81)$$

As regards the quadratic fluctuations around the j topological solutions we have now that the single ratio of determinants transforms into

$$\left(\frac{\omega}{\pi}\right)^{1/2} (2 \sinh \omega T)^{-1/2} \left\{ \frac{\text{Det}'[-(d^2/d\tau^2) + V''[x_c(\tau)]]}{\text{Det}[-(d^2/d\tau^2) + \omega^2]} \right\}^{-1/2} \longrightarrow$$

$$\left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \left[\left\{ \frac{\text{Det}'[-(d^2/d\tau^2) + V''[x_c(\tau)]]}{\text{Det}[-(d^2/d\tau^2) + \omega^2]} \right\}^{-1/2} \right]^j \quad (82)$$

according to the limit of the factor associated with the harmonic oscillator when T is large. With all this information we can write the complete transition amplitudes for the double-well potential so that

$$\langle x_f = 1 | \exp(-HT) | x_i = -1 \rangle = \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \sum_{j=1}^{\infty} \frac{(\omega T d)^{2j+1}}{(2j+1)!} \quad (83)$$

where d stands for the so-called instanton density given by

$$d = \sqrt{\frac{6}{\pi}} \sqrt{S_{eo}} \exp(-S_{eo}) \quad (84)$$

To sum up

$$\langle x_f = 1 | \exp(-HT) | x_i = -1 \rangle = \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \sinh(\omega T d) \quad (85)$$

Similarly

$$\langle x_f = 1 | \exp(-HT) | x_i = 1 \rangle = \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \sum_{j=0}^{\infty} \frac{(\omega T d)^{2j}}{(2j)!} \quad (86)$$

so that

$$\langle x_f = 1 | \exp(-HT) | x_i = 1 \rangle = \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \cosh(\omega T d) \quad (87)$$

Taking the limit $T \rightarrow \infty$ in (85), we obtain the energy eigenvalues E_0 and E_1 of the first two levels of the double-well potential

$$E_0 \approx \frac{\omega}{2} - 2\omega \sqrt{\frac{\omega}{\pi}} \exp(-2\omega/3) \quad (88)$$

$$E_1 \approx \frac{\omega}{2} + 2\omega \sqrt{\frac{\omega}{\pi}} \exp(-2\omega/3) \quad (89)$$

As anticipated in Section III the quantum mechanical tunneling transfers the wave function from one well to the other, thus lifting the degeneracy of the classical vacua. To close, notice the way in which the energy eigenvalues depend on the barrier-penetration factor, i.e. the exponential of minus the classical action of the instanton.

V. APPENDIX A.

In this appendix we include some results concerning the shape-invariance symmetry in one-dimensional supersymmetric quantum mechanics (susy qm for short). First of all one can assume the existence of two first-order differential operators A , A^\dagger which in terms of the superpotential function $W(z)$ are of the form

$$A = \frac{d}{dz} + W(z) \quad (90)$$

$$A^\dagger = -\frac{d}{dz} + W(z) \quad (91)$$

This enables us to write now a couple of hamiltonians H_- and H_+ given by

$$H_- = A^\dagger A \quad (92)$$

$$H_+ = A A^\dagger \quad (93)$$

To be precise

$$H_\pm = -\frac{d^2}{dz^2} + V_\pm(z) \quad (94)$$

for

$$V_\pm(z) = W^2(z) \pm W'(z) \quad (95)$$

where the prime denotes as usual the derivative with respect to the spatial coordinate. It is often the case to refer to the pair of potentials $V_-(z)$ and $V_+(z)$ as supersymmetric partners in accordance with the standard susy qm language. Now we look for the supersymmetric form of a standard quantum mechanical model associated with the hamiltonian H_V given by

$$H_V = -\frac{d^2}{dz^2} + V(z) \quad (96)$$

where $V(z)$ is some well-behaved function such that H_V has a non-empty discrete spectrum which is below a possible continuous spectrum. It seems plausible to identify H_V with the hamiltonian H_- of a standard susy qm model. Being ϵ the energy of the ground state $\phi_o(z)$ of H_V we can write that

$$H_V = H_- + \epsilon \quad (97)$$

so that ϵ represents the role of the factorization energy while the superpotential $W(z)$ which appears in the first differential operators A and A^\dagger is given by [6]

$$W(z) = -\frac{\phi'_o(z)}{\phi_o(z)} \quad (98)$$

In other words, we are dealing with a model where the ground state of H_V is nothing but the zero-mode of H_- . Once written H_+ in the usual way, the two partner hamiltonians H_- and H_+ are essential isospectral: the ground state energy of H_- vanishes and all other eigenvalues coincide with that of H_+ . In such a case the hamiltonian \tilde{H}_V , defined as

$$\tilde{H}_V = H_+ + \epsilon \quad (99)$$

exhibits a spectrum which coincides with the set of energy eigenvalues of H_V except for ϵ itself. It is obvious that by repetition of this method we can find a hierarchy of essential isospectral hamiltonians.

At this point we can expose in brief how the so-called shape-invariance property allows to solve in a closed form the Schrodinger's equation. Let us assume the existence of a susy qm system described in terms of a superpotential $W(z, a_o)$ which depends on the parameter a_o . The partner potentials $V_\pm(z, a_o)$ given by

$$V_\pm(z, a_o) = W^2(z, a_o) \pm W'(z, a_o) \quad (100)$$

are called shape-invariant if they are related by

$$V_+(z, a_o) = V_-(z, a_1) + R(a_1) \quad (101)$$

where a_1 is a new parameter written in terms of a_o as $a_1 = F(a_o)$ while the residual contribution $R(a_1)$ is in fact independent of the variable z . To sum up, the shape-invariance property means that the partner potential $V_+(z, a_o)$ can be considered as a new potential $V_-(z, a_1)$ with superpotential $W(z, a_1)$ once the $R(a_1)$ has been subtracted. If the mapping $a_s = F(a_{s-1})$ may be iterated leading to a family of well-behaved superpotentials $W(z, a_s)$, then the energy eigenvalues and their corresponding wave functions can be obtained in a simple way [6].

As an example we focus our interest on the set of hamiltonians O_ℓ given by

$$O_\ell = -\frac{d^2}{dz^2} - \frac{\ell(\ell+1)}{\cosh^2 z} + \ell^2 \quad (102)$$

where $\ell = 1, 2, \dots$, although only the first members of the series are relevant in physics. It is the case that O_ℓ can be factorized in terms of a superpotential $W(z, \ell)$ like

$$W(z, \ell) = \ell \tanh z \quad (103)$$

so that $O_\ell = A_\ell^\dagger A_\ell$ (notice that the energy of factorization ϵ vanishes) for

$$A_\ell = \frac{d}{dz} + \ell \tanh z \quad (104)$$

$$A_\ell^\dagger = -\frac{d}{dz} + \ell \tanh z \quad (105)$$

The shape-invariance condition appears once we write the partner hamiltonian \tilde{O}_ℓ so that in this example the mapping between the old parameter ℓ and the new one $\tilde{\ell}$ reduces to

$$\tilde{\ell} = \ell - 1 \quad (106)$$

The procedure may be iterated leading to a family of well-behaved superpotentials which allow us to extract the physical content of O_ℓ . The discrete spectrum includes a normalizable zero-energy mode $\phi_{\ell,o}(z)$ of the form

$$\phi_{\ell,o}(z) = \frac{\sqrt{2(2\ell-1)!}}{2^\ell(\ell-1)!} \frac{1}{\cosh^\ell z} \quad (107)$$

which is below the set of states $\phi_{\ell,m}(z)$ given by

$$\phi_{\ell,m}(z) = \frac{\sqrt{2(2\ell - 2m - 1)!}}{2^{\ell-m}(\ell - m - 1)!} \frac{1}{\sqrt{\prod_{j=0}^{m-1}(E_m - E_j)}} A_{\ell}^{\dagger}(z) \cdots A_{\ell-m+1}^{\dagger}(z) \left[\frac{1}{\cosh^{\ell-m} z} \right] \quad (108)$$

with energies

$$E_m = \ell^2 - (\ell - m)^2 \quad (109)$$

for $m = 1, \dots, \ell - 1$. The continuous spectrum $\phi_{\ell,k}(z)$ corresponds to

$$\phi_{\ell,k}(z) = \frac{A_{\ell}^{\dagger}(z)}{\sqrt{k^2 + \ell^2}} \frac{A_{\ell-1}^{\dagger}(z)}{\sqrt{k^2 + (\ell - 1)^2}} \cdots \frac{A_1^{\dagger}(z)}{\sqrt{k^2 + 1}} \left[\frac{\exp(ikz)}{\sqrt{2\pi}} \right] \quad (110)$$

with energy eigenvalues

$$E_k = k^2 + \ell^2 \quad (111)$$

and standard normalization as follows

$$\int_{-\infty}^{\infty} \phi_{\ell,k}^*(z) \phi_{\ell,k'}(z) dz = \delta(k - k') \quad (112)$$

VI. APPENDIX B.

The first quantum correction to the classical term in the path-integral formalism is calculated by evaluating the determinant of a differential operator. As the determinant includes the product of the eigenvalues at issue the result would be in principle a terribly divergent expression. As far as we know Hawking [2] was the first to consider in a systematic way the zeta-function method as a useful technique for the study of these determinants. For reference, let us take a one-dimensional hamiltonian H with positive discrete eigenvalues a_j . If the corresponding eigenfunctions are $\phi_j(z)$ one can write that

$$H\phi_j(z) = a_j\phi_j(z) \quad (113)$$

Next we introduce the so-called zeta-function associated with H , i.e.

$$\zeta_H(s) = \sum_j \frac{1}{a_j^s} \quad (114)$$

Now if one notes that

$$\frac{d\zeta_H(s)}{ds} = - \sum_j \ln a_j \exp(-s \ln a_j) \quad (115)$$

the determinant of H is given by

$$\text{Det } H = \exp -\zeta'_H(0) \quad (116)$$

This structure is enriched by the existence of the heat-kernel function $G(z, w, \mu)$, i.e.

$$G(z, w, \mu) = \sum_j \exp(-a_j\mu) \phi_j^*(z) \phi_j(w) \quad (117)$$

which satisfies the heat diffusion equation

$$H_z G(z, w, \mu) = - \frac{\partial G(z, w, \mu)}{\partial \mu} \quad (118)$$

with an initial condition like

$$G(z, w, 0) = \delta(z - w) \quad (119)$$

On the other hand $\zeta_H(s)$ can be understood as the Mellin transformation of $G(z, w, \tau)$ so that

$$\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mu^{s-1} d\mu \int_{-\infty}^\infty G(z, z, \mu) dz \quad (120)$$

To sum up, if we know the eigenfunctions $\phi_j(z)$ and the eigenvalues a_j of the hamiltonian H it suffices to insert such information in (117) to compute $\zeta_H(s)$. Finally we use (116) to obtain $\text{Det } H$. This zeta-function method makes it simple to study the scaling of the functional determinants we are involved with. Under a transformation of the form

$$H \rightarrow \tilde{H} = \beta H \quad (121)$$

where β represents a constant factor, one finds that

$$\text{Det } \tilde{H} = \beta^{\zeta_H(0)} \text{Det } H \quad (122)$$

as can be seen by inspection. However, this is not yet the end of the story since we need to adapt this method to the general case where the operator H also includes a continuous spectrum. To fix the ideas let us now sketch the way in which one can calculate the ratio of the determinants associated with O_ℓ (see appendix A) and P_ℓ which is given by

$$P_\ell = -\frac{d^2}{dz^2} + \ell^2 \quad (123)$$

Notice that P_ℓ itself represents a free-particle comparison hamiltonian for O_ℓ . To sum up we need to study

$$Q_\ell = \frac{\text{Det}' O_\ell}{\text{Det } P_\ell} \quad (124)$$

where $\text{Det}' O_\ell$ denotes the reduced determinant obtained once the zero-mode has been explicitly removed. To handle the continuous spectrum we resort to the density matrix $\Xi_{O_\ell}(k, z, w)$ written as [9]

$$\Xi_{O_\ell}(k, z, w) = \phi_{\ell,k}^*(z) \phi_{\ell,k}(w) \quad (125)$$

When going to P_ℓ we get

$$\Upsilon_{P_\ell}(k, z, w) = \frac{\exp[-ik(z-w)]}{2\pi} \quad (126)$$

as corresponds to a free-particle model. An additional subtraction prescription leads us to the regularized spectral density $\rho_r(k)$ expressed as

$$\rho_r(k) = \int_{-\infty}^{\infty} [\Xi_{O_\ell}(k, z, z) - \Upsilon_{P_\ell}(k, z, z)] dz \quad (127)$$

In this framework the appropriate zeta-function for the analysis of (124) would be

$$\zeta_r(s) = \zeta_{O_\ell}(s) - \zeta_{P_\ell}(s) \quad (128)$$

so that

$$\zeta_r(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mu^{s-1} d\mu \left\{ \exp \left[- \sum_{m=1}^{\ell-1} [\ell^2 - (\ell - m)^2] \mu \right] + \int_{-\infty}^{\infty} \rho_r(k) \exp[-(k^2 + \ell^2)\mu] dk \right\} \quad (129)$$

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- [1] A. Polyakov, Nucl. Phys. **B120** (1977) 429.
- [2] S.W. Hawking, Commun. Math. Phys. **55** (1977) 133.
- [3] H. Kleinert, *Paths Integrals in Quantum Mechanics, Statistics and Polymer Physics*. Singapore: World Scientific (1990).
- [4] P. Ramond, *Field Theory: A Modern Primer*. New York: Addison-Wesley Publishing Company (1992).
- [5] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*. New York: Academic Press (1965).
- [6] G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics*. Berlin: Springer-Verlag (1996).
- [7] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*. New York: Dover Publications (1970).
- [8] M.A. Shifman, *ITEP Lectures on Particle Physics and Field Theory*. Singapore: World Scientific (1999).
- [9] A.J. Niemi and G.W. Semenoff, Phys. Rep. **135** (1986) 99.